AN ASYMPTOTIC VARIANT OF THE FUBINI THEOREM FOR MAPS INTO CAT(0)-SPACES

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ABSTRACT. The classical Fubini theorem asserts that the multiple integral is equal to the repeated one for any integrable function on a product measure space. In this paper, we derive an asymptotic variant of the Fubini theorem for maps into CAT(0)-spaces from the L^1 and L^2-concentration of the maps.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The classical Fubini theorem asserts that the multiple integral is equal to the repeated one for any integrable function on a product measure space. In this paper, we prove an asymptotic variant of the Fubini theorem for maps into CAT(0)-spaces.

For this purpose, let us define the expectation (integral) for a map from a probability space into a CAT(0)-space. Throughout this section, let N be a CAT(0)-space. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a Probability space. For an N-valued random variable $Z : \Omega \to N$ such that the push-forward measure $Z_*\mathbb{P}$ has finite moment of order 1, we define its *expectation* $\mathbb{E}(Z)$ as the center of mass of the measure $Z_*\mathbb{P}$ (see Subsection 2.2 for the definition of the center of mass). This definition of expectations is based on the classical point of view of [4]. In [4], C. F. Gauss defined the expectations of random variables with values in Euclidean spaces as the above way. In the context of metric spaces, this point of view was successfully used by [1], [8], [16], and many others.

Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be two mm-spaces. Here, an *mm-space* is a triple (X, d_X, μ_X) of a set X, a complete separable distance function d_X on X, and a Borel probability measure μ_X on (X, d_X) with full-support. The *product mm-space* of X and Y is the mm-space $X \times Y$ equipped with the ℓ_2-distance function and the product probability measure. For a Borel measurable map $f : X \times Y \to N$ such that the push-forward measure $f_*(\mu_X \times \mu_Y)$ has finite moment of order 1 and $y \in Y$, we shall consider the map $f^y : X \to N$ defined by $f^y(x) := f(x, y)$. Note that the push-forward measure $(f^y)_*(\mu_X)$ has finite moment of order 1 for μ_Y-a.e. $y \in Y$. For defining the repeated integral for the map f, we assume the following:

(1) The map $g_f : Y \to N$ defined by $g_f(y) := \mathbb{E}(f^y)$ is Borel measurable.

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(2) The push-forward measure $(g_f)_*(\mu_Y)$ have finite moment of order 1.

If the map f is uniformly continuous, then the map g_f satisfies the above (1) and (2) (see Lemmas 3.1 and 3.2). It seems that the above (1) and (2) hold for an arbitrary Borel measurable map f, but the author does not know how to prove it as of now. For a Borel measurable map f satisfying the above (1) and (2), we define its *repeated integral* $\mathbb{E}_y(\mathbb{E}(f^y))$ by the expectation $\mathbb{E}(g_f)$. We will see that the Fubini theorem $\mathbb{E}(f) = \mathbb{E}_y(\mathbb{E}(f^y))$ does not hold in general for a nonlinear CAT(0)-space N (see Example 3.3). However, we succeed to estimate their difference $d_N(\mathbb{E}(f), \mathbb{E}_y(\mathbb{E}(f^y)))$ by the term of the L^1 and L^2-variation of the map f: Let X be an mm-space and $p \geq 1$. Given a Borel measurable map $f : X \to N$, we define its L^p*-variation* by

$$V_p(f) := \Big(\int\int_{X\times X} d_N(f(x), f(x'))^p d\mu_X(x) d\mu_X(x') \Big)^{1/p}.$$

A main theorem of this paper is the following:

Theorem 1.1. *Let X and Y be two mm-spaces. Then, for any uniformly continuous map $f : X \times Y \to N$ such that the push-forward measure $f_*(\mu_X \times \mu_Y)$ has the finite moment of order 1, we have*

$$d_N(\mathbb{E}(f), \mathbb{E}_y(\mathbb{E}(f^y))) \leq V_1(f) \tag{1.1}$$

and

$$d_N(\mathbb{E}(f), \mathbb{E}_y(\mathbb{E}(f^y))) \leq \frac{1}{\sqrt{3}} V_2(f) \tag{1.2}$$

Jensen's inequality easily leads to the inequality (1.1). In the proof of the inequality (1.2), we iterate some K-T. Sturm's inequality about the center of mass of a probability measure on a CAT(0)-space (see Proposition 2.8). We emphasize that the coefficient $1/\sqrt{3}$ of the inequality (1.2) cannot be obtained only from the inequalities (1.1) and $V_1(f) \leq V_2(f)$.

Let $\{X_n\}_{n=1}^\infty$ be a sequence of mm-spaces and $\{N_n\}_{n=1}^\infty$ a sequence of CAT(0)-spaces. For $p \geq 1$, we say that a sequence $\{f_n : X_n \to N_n\}_{n=1}^\infty$ of Borel measurable maps L^p-*concentrates* if $V_p(f_n) \to 0$ as $n \to \infty$. From the inequality (1.1), the L^p-concentration of uniformly continuous maps implies that the Fubini theorem for the maps "almostly" holds. The L^2-concentration theory of maps into CAT(0)-spaces was first studied by M. Gromov in [5]. In [2], the author also studied relationships between the Lévy-Milman concentration theory of 1-Lipschitz maps and the L^p-concentration theory of 1-Lipschitz maps (see [10], [11], [12], [13], and [15] for further information about the Lévy-Milman concentration theory). Motivated by Gromov's works in [5], [6], and [7], the author studied the L^p-concentration theory of 1-Lipschitz maps into Hadamard manifolds and $\mathbb{R}$-trees in [2] and [3]. Combining Theorem 1.1 with author's works and Gromov's works, we obtain the following corollary: We shall consider each compact connected Riemannian manifold M as an mm-space equipped with the volume measure normalized to have the total volume 1. We denote by $\lambda_1(M)$ the non-zero first eigenvalue of the Laplacian on M.

Corollary 1.2. *Let M be a compact Riemannian manifold. Then, for any n-dimensional Hadamard manifold N' and 1-Lipschitz map $f : M \times M \to N'$, we have*

$$d_{N'}(\mathbb{E}(f), \mathbb{E}_y(\mathbb{E}(f^y))) \leq 2\sqrt{\frac{2n}{3\lambda_1(M)}}.$$

For an $\mathbb{R}$-tree T and a 1-Lipschitz map $f : M \times M \to T$, we also have

$$d_T(\mathbb{E}(f), \mathbb{E}_y(\mathbb{E}(f^y)))^2 \leq \frac{8(38+16\sqrt{2})}{3\lambda_1(M)}.$$

2. Preliminaries

2.1. **The Wasserstein distance function of order 1.** Let (X, d_X) be a complete metric space. For $p \geq 1$, we indicate by $\mathcal{P}_p(X)$ the set of all probability measures ν such that ν has the separable support and $\int_X d_X(x,y)^p \, d\nu(y) < +\infty$ for some (hence all) $x \in X$.

For $\mu, \nu \in \mathcal{P}_1(X)$, we define the *Wasserstein distance* $d_1^W(\mu,\nu)$ *of order* 1 between μ and ν as the infimum of $\int_{X\times X} d_X(x,y) \, d\pi(x,y)$, where $\pi \in \mathcal{P}_1(X \times X)$ runs over all *couplings* of μ and ν, that is, the probability measures π with the property that $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for any Borel subset $A \subseteq X$.

Theorem 2.1 (L. V. Kantorovich, cf. [17, Theorem 5.1, Remark 6.5])**.** *For any $\mu, \nu \in \mathcal{P}_1(X)$, we have*

$$d_1^W(\mu,\nu) = \sup\Big\{ \int_X \psi(x) d\mu(x) - \int_X \psi(x) d\nu(x) \Big\},$$

where the supremum is taken over all 1-Lipschitz function $\psi : X \to \mathbb{R}$.

2.2. **Basics of the center of mass of a measure on CAT(0)-spaces.** In this subsection, we review Sturm's works about probability measures on a CAT(0)-spaces, which is needed for the proof of the main theorem. Refer [9] and [16] for details.

We shall recall some standard terminologies in metric geometry. Let (X, d_X) be a metric space. A rectifiable curve $\gamma : [0,1] \to X$ is called a *geodesic* if its arclength coincides with the distance $d_X(\gamma(0), \gamma(1))$ and it has a constant speed, i.e., parameterized proportionally to the arc length. We say that (X, d_X) is a *geodesic metric space* if any two points in X are joined by a geodesic between them. A geodesic metric space N is called a *CAT(0)-space* if we have

$$d_N(x, \gamma(1/2))^2 \leq \frac{1}{2} d_N(x,y)^2 + \frac{1}{2} d_N(x,z)^2 - \frac{1}{4} d_N(y,z)^2$$

for any $x, y, z \in N$ and any minimizing geodesic $\gamma : [0,1] \to N$ from y to z. For example, Hadamard manifolds, Hilbert spaces, and $\mathbb{R}$-trees are all CAT(0)-spaces.

For any $\nu \in \mathcal{P}_1(X)$ and $z \in X$, we consider the function $h_{z,\nu} : X \to \mathbb{R}$ defined by

$$h_{z,\nu}(x) := \int_X \{ d_X(x,y)^2 - d_X(z,y)^2 \} d\nu(y).$$

Note that

$$\int_X |\, d_X(x,y)^2 - d_X(z,y)^2 | d\nu(y) \le d_X(x,z) \int_X \{d_X(x,y) + d_X(z,y)\} d\nu(y) < +\infty.$$

A point $z_0 \in X$ is called the *center of mass* of the measure $\nu \in \mathcal{B}_1(X)$ if for any $z \in X$, z_0 is a unique minimizing point of the function $h_{z,\nu}$. We denote the point z_0 by $c(\nu)$. Note that if the measure ν moreover satisfies that $\nu \in \mathcal{P}_2(X)$, then we have

$$\int_X d_X(c(\nu),y)^2 \ d\nu(y) = \inf_{x\in X} \int_X d_X(x,y)^2 \ d\nu(y).$$

A metric space X is said to be *centric* if every $\nu \in \mathcal{P}_1(X)$ has the center of mass.

Proposition 2.2 (cf. [16, Proposition 4.3])**.** *A CAT(0)-space is centric.*

A simple variational argument implies the following lemma:

Lemma 2.3 (cf. [16, Proposition 5.4])**.** *Let H be a Hilbert space. Then, for each $\nu \in \mathcal{P}_1(H)$, we have*

$$c(\nu) = \int_H y d\nu(y).$$

Lemma 2.4 (cf. [16, Proposition 5.10])**.** *Let N be a Hadamard manifold and $\nu \in \mathcal{P}_1(N)$. Then, $x = c(\nu)$ if and only if*

$$\int_N \exp_x^{-1}(y) d\nu(y) = 0.$$

In particular, identifying the tangent space of N at $c(\nu)$ with the Euclidean space of the same dimension, we have $c((\exp_{c(\nu)}^{-1})_*(\nu)) = 0$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and N a centric metric space. For an N-valued random variable $Z : \Omega \to N$ satisfying $Z_*\mathbb{P} \in \mathcal{P}_1(N)$, we define its *expectation* $\mathbb{E}(Z) \in N$ by the point $c(Z_*\mathbb{P})$.

Let X be a geodesic metric space. A function $\varphi : X \to \mathbb{R}$ is called *convex* if the function $\varphi \circ \gamma : [0,1] \to \mathbb{R}$ is convex for each geodesic $\gamma : [0,1] \to X$.

Proposition 2.5 (Convexity of a distance function, cf. [16, Corollary 2.5])**.** *Let N be a CAT(0)-space and $\gamma, \eta : [0,1] \to N$ be two geodesics. Then, for any $t \in [0,1]$, we have*

$$d_N(\gamma(t), \eta(t)) \le (1-t)\ d_N(\gamma(0), \eta(0)) + t\ d_N(\gamma(1), \eta(1)).$$

Theorem 2.6 (Jensen's inequality, cf. [16, Theorem 6.2])**.** *Let N be a CAT(0)-space. Then, for any lower semicontinuous convex function $\varphi : N \to \mathbb{R}$ and $\nu \in \mathcal{P}_1(N)$, we have*

$$\varphi(c(\nu)) \le \int_N \varphi(x) \ d\nu(x),$$

provided the right-hand side is well-defined.

Applying Proposition 2.5 to Theorem 2.6, we obtain the following corollary:

Corollary 2.7. *Let N be a CAT(0)-space. Then, for any $p_0 \in N$ and $\nu \in \mathcal{P}_1(N)$, we have*

$$d_N(p_0, c(\nu)) \leq \int_N d_N(p_0, p) d\nu(p).$$

Proposition 2.8 (Variance inequality, [16, Proposition 4.4])**.** *Let N be a CAT(0)-space and $\nu \in \mathcal{P}_1(N)$. Then, for any $z \in N$, we have*

$$\int_N \{d_N(z,x)^2 - d_N(c(\nu), x)^2\} d\nu(x) \geq d_N(z, c(\nu))^2. \tag{2.1}$$

Note that if N is a Hilbert space, then we have the equality in (2.1).

Proposition 2.9 (cf. [16, Theorem 2.5])**.** *Let N be a CAT(0)-space. Then, for any $\mu, \nu \in \mathcal{P}_1(N)$, we have $d_N(c(\mu), c(\nu)) \leq d_1^W(\mu, \nu)$.*

3. Proof of the main theorem

Let X and Y be a two mm-spaces and N a CAT(0)-space. Given a uniformly continuous map $f : X \times Y \to N$ with $f_*(\mu_X \times \mu_Y) \in \mathcal{P}_1(N)$, we easily see that $(f^y)_*(\mu_X) \in \mathcal{P}_1(N)$ for μ_Y-a.e. $y \in Y$. Since Y has the full-support and the map f is uniformly continuous, we see that $(f^y)_*(\mu_X) \in \mathcal{P}_1(N)$ for any $y \in N$. We shall consider the map $g_f : Y \to N$ defined by $g_f(y) := \mathbb{E}(f^y)$.

Lemma 3.1. *The map $g_f : Y \to N$ is uniformly continuous. In particular, the map is Borel measurable.*

Proof. From Theorem 2.1 and Proposition 2.6, for any $y, y' \in Y$, we have

$$\begin{aligned} d_N(g_f(y), g_f(y')) \leq\ & d_1^W((f^y)_*(\mu_X), (f^{y'})_*(\mu_X)) \\ =\ & \sup\Big\{ \int_N \psi(z) d(f^y)_*(\mu_X)(z) - \int_N \psi(z) d(f^{y'})_*(\mu_X)(z) \Big\} \\ =\ & \sup\Big\{ \int_X \psi(f(x,y)) d\mu_X(x) - \int_X \psi(f(x,y')) d\mu_X(x) \Big\} \\ \leq\ & \int_X d_N(f(x,y), f(x,y')) d\mu_X(x), \end{aligned}$$

where each supremum is taken over all 1-Lipschitz function $\psi : N \to \mathbb{R}$. Observe that the right-hand side of the above inequality converges to zero as $d_Y(y, y') \to 0$. This completes the proof. ◻

Lemma 3.2. *We have $(g_f)_*(\mu_Y) \in \mathcal{P}_1(N)$.*

Proof. Taking any point $p_0 \in N$, from Corollary 2.7, we obtain

$$\int_Y d_N(\mathbb{E}(f^y), p_0) d\mu_Y(y) \leq \int_{X \times Y} d_N(f(x,y), p_0) d(\mu_X \times \mu_Y)(x,y) < +\infty.$$

This completes the proof. ◻

The following example asserts that the equality $\mathbb{E}(f) = \mathbb{E}_y(\mathbb{E}(f^y))$ does not hold for non-linear CAT(0)-spaces in general:

Example 3.3. For $i = 1, 2, 3$, let $T_i := \{(i, r) \mid r \in [0, +\infty)\}$ be a copy of $[0, +\infty)$ equipped with the usual Euclidean distance function. The *tripod* T is the metric space obtained by gluing together all these spaces T_i, $i = 1, 2, 3$, at their origins with the intrinsic distance function. Let $\{a, b\}$ be an arbitrary two-point mm-space equipped with the uniform probability measure. Let us consider the map $f : \{a, b\}^2 \to T$ defined by $f(a, a) := (1, 1) \in T_1$, $f(b, a) := (2, 1) \in T_2$, and $f(a, b) = f(b, b) := (3, 1) \in T_3$. In this case, we easily see that $\mathbb{E}(f) = (0, 0)$, $\mathbb{E}(f^a) = (0, 0)$, $\mathbb{E}(f^b) = (3, 1)$, and therefore $\mathbb{E}_y(\mathbb{E}(f^y)) = (3, 1/2)$.

Proof of Theorem 1.1. Iterating Corollary 2.7, we have

$$\begin{aligned}
d_N(\mathbb{E}(f), \mathbb{E}_y(\mathbb{E}(f^y))) &\leq \int_{X \times Y} d_N(f(x, y'), \mathbb{E}_y(\mathbb{E}(f^y))) d(\mu_X \times \mu_Y)(x, y') \\
&\leq \int_{X \times Y \times Y} d_N(f(x, y'), \mathbb{E}(f^{y''})) d(\mu_X \times \mu_Y \times \mu_Y)(x, y', y'') \\
&\leq V_1(f).
\end{aligned}$$

Thereby, we obtain the inequality (1.1).

To prove the inequality (1.2), we are going to iterate Proposition 2.8. Since $f_*(\mu_X \times \mu_Y) \notin \mathcal{P}_2(N)$ implies $V_2(f) = +\infty$, we assume that $f_*(\mu_X \times \mu_Y) \in \mathcal{P}_2(N)$. From Proposition 2.8, we have

$$\begin{aligned}
(3.1) \quad & \int_{X \times Y} d_N(f(x, y'), \mathbb{E}(f))^2 d(\mu_X \times \mu_Y)(x, y') \\
&= \int_Y d\mu_Y(y') \int_X d_N(f^{y'}(x), \mathbb{E}(f))^2 d\mu_X(x) \\
&\geq \int_Y d\mu_Y(y') \Big\{ \int_X d_N(f^{y'}(x), \mathbb{E}(f^{y'}))^2 d\mu_X(x) + d_N(\mathbb{E}(f^{y'}), \mathbb{E}(f))^2 \Big\} \\
&= \int_{X \times Y} d_N(f^{y'}(x), \mathbb{E}(f^{y'}))^2 d(\mu_X \times \mu_Y)(x, y') + \int_Y d_N(\mathbb{E}(f^{y'}), \mathbb{E}(f))^2 d\mu_Y(y') \\
&\geq \int_{X \times Y} d_N(f^{y'}(x), \mathbb{E}(f^{y'}))^2 d(\mu_X \times \mu_Y)(x, y') + \int_Y d_N(\mathbb{E}(f^{y'}), \mathbb{E}_y(\mathbb{E}(f^y)))^2 d\mu_Y(y') \\
&\qquad + d_N(\mathbb{E}(f), \mathbb{E}_y(\mathbb{E}(f^y)))^2.
\end{aligned}$$

Since

$$d_N(f^{y'}(x), \mathbb{E}(f^{y'}))^2 + d_N(\mathbb{E}(f^{y'}), \mathbb{E}_y(\mathbb{E}(f^y)))^2 \geq \frac{1}{2}\, d_N(f^{y'}(x), \mathbb{E}_y(\mathbb{E}(f^y)))^2,$$

substituting this into the inequality (3.1), we get

$$\begin{aligned}&\int_{X\times Y} d_N(f(x,y'),\mathbb{E}(f))^2 d(\mu_X\times\mu_Y)(x,y')\\ \geq\ &\frac{1}{2}\int_{X\times Y} d_N(f(x,y'),\mathbb{E}_y(\mathbb{E}(f^y)))^2 d(\mu_X\times\mu_Y)(x,y') + d_N(\mathbb{E}(f),\mathbb{E}_y(\mathbb{E}(f^y)))^2\end{aligned}$$

Since

$$\begin{aligned}&\int_{X\times Y} d_N(f(x,y'),\mathbb{E}(f))^2 d(\mu_X\times\mu_Y)(x,y')\\ \leq\ &\int_{X\times Y} d_N(f(x,y'),\mathbb{E}_y(\mathbb{E}(f^y)))^2 d(\mu_X\times\mu_Y)(x,y'),\end{aligned}$$

we therefore obtain

$$d_N(\mathbb{E}(f),\mathbb{E}_y(\mathbb{E}(f^y)))^2 \leq \frac{1}{2}\int_{X\times Y} d_N(f(x,y'),\mathbb{E}_y(\mathbb{E}(f^y)))^2 d(\mu_X\times\mu_Y)(x,y'). \tag{3.2}$$

By virtue of Proposition 2.8, we also get

$$\begin{aligned}&\int_{X\times Y} d_N(f(x,y'),\mathbb{E}_y(\mathbb{E}(f^y)))^2 d(\mu_X\times\mu_Y)(x,y')\\ \leq\ &\int_{X\times Y} d(\mu_X\times\mu_Y)(x,y')\Big\{\int_Y \big\{ d_N(f^{y'}(x),\mathbb{E}(f^{y''}))^2 - d_N(\mathbb{E}(f^{y'}),\mathbb{E}_y(\mathbb{E}(f^y)))^2\big\} d\mu_Y(y'')\Big\}\\ =\ &\int_{X\times Y\times Y} d_N(f^{y'}(x),\mathbb{E}(f^{y''}))^2 d(\mu_X\times\mu_Y\times\mu_Y)(x,y',y'')\\ &\qquad\qquad - \int_Y d_N(\mathbb{E}(f^{y'}),\mathbb{E}_y(\mathbb{E}(f^y)))^2 d\mu_Y(y').\end{aligned}\tag{3.3}$$

Since

$$d_N(f^{y'}(x),\mathbb{E}(f^{y''}))^2 \leq \int_X \big\{ d_N(f^{y''}(x'),f^{y'}(x))^2 - d_N(f^{y''}(x'),\mathbb{E}(f^{y''}))^2\big\} d\mu_X(x')$$

from Proposition 2.8, substituting this into (3.3), we have

$$\begin{aligned}&\int_{X\times Y} d_N(f(x,y'),\mathbb{E}_y(\mathbb{E}(f^y)))^2 d(\mu_X\times\mu_Y)(x,y')\\ \leq\ &V_2(f)^2 - \int_{X\times Y} d_N(f^{y'}(x),\mathbb{E}(f^{y'}))^2 d(\mu_X\times\mu_Y)(x,y') - \int_Y d_N(\mathbb{E}(f^{y'}),\mathbb{E}_y(\mathbb{E}(f^y)))^2 d\mu_Y(y')\\ \leq\ &V_2(f)^2 - \frac{1}{2}\int_{X\times Y} d_N(f^{y'}(x),\mathbb{E}_y(\mathbb{E}(f^y)))^2 d(\mu_X\times\mu_Y)(x,y').\end{aligned}$$

We therefore obtain

$$\int_{X\times Y} d_N(f(x,y'),\mathbb{E}_y(\mathbb{E}(f^{y'})))^2 d(\mu_X\times\mu_Y)(x,y') \leq \frac{2}{3}V_2(f)^2.$$

Combining this with the inequality (3.2), we finally obtain the inequality (1.2). This completes the proof. ☐

4. Applications

4.1. **Product inequalities.**

Proposition 4.1 (Y. G. Reshetnyak, cf. [16, Proposition 2.4])**.** *For any four points x_1, x_2, x_3, x_4 in a CAT(0)-space N, we have*

$$d_N(x_1,x_3)^2 + d_N(x_2,x_4)^2 \leq d_N(x_1,x_2)^2 + d_N(x_2,x_3)^2 + d_N(x_3,x_4)^2 + d_N(x_4,x_1)^2.$$

Given an mm-space X and a metric space Y we define

$$\mathrm{Obs}L^p\text{-}\mathrm{Var}_Y(X) := \sup\{V_p(f) \mid f : X \to Y \text{ is a 1-Lipschitz map}\},$$

and call it the *observable L^p-variation* of X. The idea of the observable L^p-variation comes from the quantum and statistical mechanics, that is, we think of μ_X as a state on a configuration space X and f is interpreted as an observable.

Corollary 4.2. *Let X and Y be two mm-spaces and N a CAT(0)-space. Then, we have*

$$\mathrm{Obs}L^2\text{-}\mathrm{Var}_N(X\times Y)^2 \leq \mathrm{Obs}L^2\text{-}\mathrm{Var}_N(X) + \mathrm{Obs}L^2\text{-}\mathrm{Var}_N(Y). \tag{4.1}$$

Proof. Let $f : X\times Y \to N$ be an arbitrary 1-Lipschitz map. Then, putting $Z := X \times Y$, from Proposition 4.1, we obtain

$$\begin{aligned}
V_2(f)^2 &= \frac{1}{2}\int_{Z\times Z} \{d_N(f(x,y), f(x',y'))^2 + d_N(f(x,y'), f(x',y))^2\} d(\mu_Z\times\mu_Z)(x,y,x',y') \\
&\leq \frac{1}{2}\int_{Z\times Z} \{d_N(f(x,y), f(x',y))^2 + d_N(f(x',y), f(x',y'))^2 \\
&\qquad + d_N(f(x',y'), f(x,y'))^2 + d_N(f(x,y'), f(x,y))^2\} d(\mu_Z\times\mu_Z)(x,y,x',y') \\
&= \int_X V_2(f^x)^2 d\mu_X(x) + \int_Y V_2(f^y)^2 d\mu_Y(y) \\
&\leq \mathrm{Obs}L^2\text{-}\mathrm{Var}_N(X)^2 + \mathrm{Obs}L^2\text{-}\mathrm{Var}_N(Y)^2.
\end{aligned}$$

This completes the proof. ☐

Lemma 4.3. *Let X and Y be two mm-spaces and Z a metric space. Then, for any $p \geq 1$, we have*

$$\mathrm{Obs}L^p\text{-}\mathrm{Var}_Z(X\times Y)^p \leq 2^{p-1}\,\mathrm{Obs}L^p\text{-}\mathrm{Var}_Z(X)^p + 2^{p-1}\,\mathrm{Obs}L^p\text{-}\mathrm{Var}_Z(Y)^p. \tag{4.2}$$

Proof. Given any 1-Lipschitz map $f : X\times Y \to Z$, putting $W := X\times Y$, we have

$$\begin{aligned}
V_p(f)^p &\leq \int_{W\times W} 2^{p-1}\{d_Z(f(x,y), f(x,y'))^p + d_Z(f(x,y'), f(x',y'))^p\} d(\mu_W\times\mu_W)(x,y,x',y') \\
&= 2^{p-1}\int_X V_p(f^x)^p d\mu_X(x) + 2^{p-1}\int_Y V_p(f^y)^p d\mu_Y(y) \\
&\leq 2^{p-1}\,\mathrm{Obs}L^p\text{-}\mathrm{Var}_Z(X)^p + 2^{p-1}\,\mathrm{Obs}L^p\text{-}\mathrm{Var}_Z(Y)^p.
\end{aligned}$$

This completes the proof. □

Note that the inequality (4.1) is sharper than the inequality (4.2) in the case where $p = 2$ and Z is a CAT(0)-space.

Combining Theorem 1.1 and Lemma 4.3 we obtain the following corollary:

Corollary 4.4. *Let $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$ be a sequences of mm-spaces and $\{N_n\}_{n=1}^\infty$ be a sequences of CAT(0)-spaces. Then, assuming that*

$$\mathrm{Obs}L^1\text{-}\mathrm{Var}_{N_n}(X_n) \to 0 \text{ as } n \to \infty \text{ and } \mathrm{Obs}L^1\text{-}\mathrm{Var}_{N_n}(Y_n) \to 0 \text{ as } n \to \infty,$$

we have

$$d_{N_n}(\mathbb{E}(f), \mathbb{E}_{y_n}(\mathbb{E}(f^{y_n}))) \to 0 \text{ as } n \to \infty$$

for any sequence $\{f_n : X_n \times Y_n \to N_n\}_{n=1}^\infty$ of 1-Lipschitz maps.

4.2. **The non-zero first eigenvalue of Laplacian and the observable L^2-variation.** Although the same method in [5] and [7] implies the following proposition, we prove it for the completeness.

Proposition 4.5 (cf. [5, Section 13], [7, Section $3\frac{1}{2}$.41])**.** *Let M be a compact connected Riemannian manifold and N' an n-dimensional Hadamard manifold. Then, we have*

$$\mathrm{Obs}L^2\text{-}\mathrm{Var}_{N'}(M) \leq 2\sqrt{\frac{n}{\lambda_1(M)}}.$$

Proof. Let $f : M \to N'$ be an arbitrary 1-Lipschitz map. We shall prove that

$$\int_M d_{N'}(f(x), \mathbb{E}(f))^2 d\mu_M(x) \leq \frac{n}{\lambda_1(M)}. \tag{4.3}$$

If the inequality (4.3) holds, then we finish the proof since

$$V_2(f) \leq 2\Big(\int_M d_{N'}(f(x), \mathbb{E}(f))^2 d\mu_M(x)\Big)^{1/2} \leq 2\sqrt{\frac{n}{\lambda_1(M)}}.$$

Suppose that

$$\int_M d_{N'}(f(x), \mathbb{E}(f))^2 d\mu_M(x) > \frac{n}{\lambda_1(M)}. \tag{4.4}$$

We identify the tangent space of N' at the point $\mathbb{E}(f)$ with the Euclidean space $\mathbb{R}^n$ and consider the map $f_0 := \exp^{-1}_{\mathbb{E}(f)} \circ f : M \to \mathbb{R}^n$. According to the hinge theorem (see [14, Chapter IV, Remark 2.6]), the map f_0 is a 1-Lipschitz map. Note that $|f_0(x)| = d_{N'}(f(x), \mathbb{E}(f))$ for any $x \in X$ because the map $\exp^{-1}_{\mathbb{E}(f)}$ is isometric on rays issuing from the point $\mathbb{E}(f)$. Hence, from the inequality (4.4), we have

$$\int_M |f_0(x)|^2 d\mu_M(x) > \frac{n}{\lambda_1(M)}.$$

Denoting by $(f_0(x))_i$ the i-th component of $f_0(x)$, we therefore see that there exists i_0 such that

$$\int_M |(f_0(x))_{i_0}|^2 d\mu_M(x) > \frac{1}{\lambda_1(M)}. \tag{4.5}$$

Note the function $(f_0)_{i_0}$ has the mean zero from Lemmas 2.3 and 2.4. Combining this with the inequality (4.5), we therefore obtain

$$\lambda_1(M) = \inf \frac{\int_M |\operatorname{grad}_x g|^2 d\mu_M(x)}{\int_M |g(x)|^2 d\mu_M(x)} < \lambda_1(M),$$

where the infimum is taken over all Lipschitz function $g : M \to \mathbb{R}$ with the mean zero. This is a contradiction. This completes the proof. □

One can obtain a similar result to Proposition 4.5 for a finite connected graph.

Theorem 4.6 (cf. [2, Proposition 5.7])**.** *Let X be an mm-space and T an $\mathbb{R}$-tree. Then, we have*

$$\mathrm{Obs}L^2\text{-}\mathrm{Var}_T(X)^2 \leq (38 + 16\sqrt{2})\, \mathrm{Obs}L^2\text{-}\mathrm{Var}_{\mathbb{R}}(X)^2.$$

Combining Proposition 4.5 with Theorem 4.6, we obtain the following corollary:

Corollary 4.7. *Let M be a compact connected Riemannian manifold and T an $\mathbb{R}$-tree. Then, we have*

$$\mathrm{Obs}L^2\text{-}\mathrm{Var}_T(M)^2 \leq \frac{4(38 + 16\sqrt{2})}{\lambda_1(M)}.$$

Proof of Corollary 1.2. The corollary follows from Theorem 1.1 together with Corollary 4.2, Proposition 4.5, and Corollary 4.7. This completes the proof. □

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